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TWO-DIMENSIONAL GAS VORTICES AND TWISTED GAS JETS

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This paper studies an invariant solution of rank one of the equations of motion of a polytropic gas that describes two-dimensional gas vortices and twisted gas jets. Flow types are classified according to the governing parameter: vortices in the form of sources and sinks, unlimited expansion, and collapse.

Key words: *invariant solution, two-dimensional gas vortices, twisted gas jet, analytical description of flow.*

1. Description of the Submodel. The equations of polytropic gas dynamics admit the Lie symmetry algebra L_{13} [1]. The optimal system of subalgebras ΘL_{13} of the algebra L_{13} was constructed in [2], and a description of all submodels of polytropic gas dynamics that admit three-dimensional symmetry algebras is given in [3, 4]. A description of 37 of these submodels that specify invariant solutions of rank one and have a normalizer of larger dimension is presented in [5]. For most of these models, the factor-systems are integrated in finite form or are reduced to finite formulas and one implicit ordinary differential equation.

The present paper studies one of the indicated submodels, whose equations are integrated by a more complex scheme.

Let us consider the three-dimensional subalgebra

$$L_{3.56} = \langle \partial_x, \partial_t, t\,\partial_t + \boldsymbol{x}\cdot\partial_{\boldsymbol{x}} + a(\rho\,\partial_\rho + p\,\partial_p)\rangle \tag{1.1}$$

from the optimal system of subalgebras ΘL_{13} of the Lie symmetry algebra L_{13} of the equations of polytropic gas dynamics (the subalgebra number corresponds to the list in [2]). The normalizer [1] of this three-dimensional subalgebra is a subalgebra of dimension seven. This suggests that the equations of the submodel have intermediate integrals. In (1.1), $\boldsymbol{x} = (x, y, z)$; $a \neq 0$ is an arbitrary parameter. We construct an invariant solution of rank one for the algebra $L_{3.56}$. This solution is conveniently represented in cylindrical coordinates with the Ox axis which are related to Cartesian coordinates by the formulas

$$\begin{gathered} x = x, \qquad y = r\sin\theta, \qquad z = r\cos\theta, \\ v = V\cos\theta - W\sin\theta, \quad w = V\sin\theta + W\cos\theta, \end{gathered} \tag{1.2}$$

where V and W are the radial and circumferential velocity components.

The third operator of algebra (1.1) in coordinates (1.2) becomes $t\,\partial_t + x\,\partial_x + r\,\partial_r + a(\rho\,\partial_\rho + p\,\partial_p)$. The invariants of the algebra $L_{3.56}$ written in cylindrical coordinates have the form

$$\theta,\ u,\ V,\ W,\ r^{-a}\rho,\ r^{-a}p, \tag{1.3}$$

where θ is the polar angle in the plane $\mathbb{R}^2(y,z)$, u is the velocity component along the Ox axis, p is the pressure, and ρ is the gas density. The equation of state has the form $p = s\rho^\gamma$, where s is an entropy function and $\gamma > 1$ is the adiabatic exponent.

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According to (1.3), the invariant solution is represented as

$$u = U(\theta), \quad V = V(\theta), \quad W = W(\theta), \quad \rho = r^a R(\theta), \quad p = r^a P(\theta), \quad s = r^{a(1-\gamma)} S, \tag{1.4}$$

where $S = PR^{-\gamma}$. Hence, expressions (1.4) are a solution with variable entropy. An isentropic solution is possible only for $\gamma = 1$ and $PR^{-1} = S_0 = \text{const}$. In the present paper, this case is not considered. The invariant functions U, V, W, R, P, and S satisfy the factor-system

$$\begin{gathered} WU' = 0, \qquad WV' + aP/R = W^2, \qquad WW' + P'/R = -VW, \\ WR' + (a+1)RV + RW' = 0, \qquad WP' + (a+\gamma)PV + \gamma PW' = 0, \\ a(1-\gamma)VS + WS' = 0 \end{gathered} \tag{1.5}$$

(the prime denotes derivatives with respect to θ). System (1.5) is invariant under the transformation $W \to -W$, $\theta \to -\theta$, which corresponds to a change in the reference direction of the polar angle. Hence, we can consider only the case $W > 0$. The last equation in system (1.5) — the equation for entropy — is a consequence of the previous two equations. This equation is given because the function S is used for the further transformation of the system. The sound velocity c is calculated by the formula $c^2 = \gamma p/\rho$. From formulas (1.4), it follows that this quantity is invariant [$c^2 = c^2(\theta) = \gamma P/R$] and satisfies the equation

$$W(c^2)' + (\gamma - 1)(V + W')c^2 = 0. \tag{1.6}$$

In the coordinate system (x, r, θ), the vortex $\boldsymbol{\omega} = \operatorname{rot} \boldsymbol{u}$ of the velocity field $\boldsymbol{u}$ given by expressions (1.4) is written as

$$\boldsymbol{\omega} = (r^{-1}(W - V'), r^{-1}U', 0). \tag{1.7}$$

Generally, $\boldsymbol{\omega} \neq 0$.

System (1.5) has a nontrivial constant solution in which the radial velocity component $V \equiv 0$ and the remaining functions in the solution are given by the formulas

$$U = U_0, \qquad W^2 = W_0^2 = ac_0^2/\gamma, \qquad R = R_0, \qquad P = P_0, \tag{1.8}$$

where P_0 and R_0 are the integration constants; $c_0^2 = \gamma P_0/R_0$. From Eq. (1.8) it follows that this solution exists only for $a > 0$. This solution corresponds to gas motion in a round tube whose streamlines are screw lines:

$$x = (r_0 U_0/W_0)\theta + x_0, \qquad r = r_0.$$

The values $\theta = \theta_0 = 0$, $r = r_0$, and $x = x_0$ specify some position of a gas particle on a streamline.

Solution (1.4), which is a new solution not studied previously, generalizes the classical two-dimensional self-similar solutions of the gas dynamics equations [6, 7] and differs from the solutions considered by Ovsyannikov [8] and from conical flows [9].

2. Transformation of System (1.5). Next, we will assume that $W \not\equiv 0$. Otherwise, from Eqs. (1.5) and (1.6) and the equation of state, it follows that $P = R = 0$. To integrate system (1.5), we use the method of straightening derivatives which was employed in [10] to study the regular partially invariant solutions of the equations of gas dynamics. For $W \not\equiv 0$, instead of θ we introduce the new independent variable $\sigma = \sigma(\theta)$ which straightens the derivative $W(d/d\theta)$, so that

$$W \frac{df}{d\theta} = \frac{df}{d\sigma}, \qquad \frac{d\sigma}{d\theta} = \frac{1}{W} \tag{2.1}$$

for any smooth function $f = f(\theta)$. We consider the second equation (2.1) as an expression for the velocity component W:

$$W = \theta_\sigma, \tag{2.2}$$

taking into account that $\theta = \theta(\sigma)$. The parameter σ, which changes along the streamlines, plays the role of time, specifying the rate of variation in the polar angle θ. This change is biunique for the sign definite function W. The value $W = 0$, which is a singular point of the solution, separates motions with $W > 0$, for which the angle θ increases monotonically, from motions for which $W < 0$ and θ decreases during gas particle motion. As noted above, system (1.5) is invariant under the involution: $W \to -W$ and $\theta \to -\theta$.

Introducing the quantity

$$q = \ln S \tag{2.3}$$

into the equation for entropy [the last equation in system (1.5)] and recalculating the derivative by formula (2.1), we express the velocity component V in terms of the derivative of the entropy:

$$V = \frac{1}{a(\gamma-1)}\, W (\ln S)' = \frac{1}{a(\gamma-1)}\, (\ln S)_\sigma = \frac{q_\sigma}{a(\gamma-1)}. \tag{2.4}$$

For $W \not\equiv 0$, the first equation of system (1.5) implies that $U = U_0 = \text{const}$. After division by R and P and substitution of expression (2.2) and (2.4), the fourth and fifth equations of this system become

$$(\ln R)_\sigma + \frac{a+1}{a(\gamma-1)}\, q_\sigma + \frac{\theta_{\sigma\sigma}}{\theta_\sigma} = 0, \qquad (\ln P)_\sigma + \frac{a+\gamma}{a(\gamma-1)}\, q_\sigma + \gamma\, \frac{\theta_{\sigma\sigma}}{\theta_\sigma} = 0. \tag{2.5}$$

As a result of integration, Eqs. (2.5) leads to the representations

$$R = R_0 \Big(\theta_\sigma\, \mathrm{e}^{\frac{a+1}{a(\gamma-1)}\, q}\Big)^{-1}, \qquad P = P_0 \Big(\theta_\sigma^\gamma\, \mathrm{e}^{\frac{a+\gamma}{a(\gamma-1)}\, q}\Big)^{-1}, \tag{2.6}$$

where R_0 and P_0 are integration constants. Equation (2.6) leads to the expression

$$\frac{P}{R} = \frac{P_0}{R_0}\, \theta_\sigma^{1-\gamma}\, \mathrm{e}^{-q/a}. \tag{2.7}$$

Substitution of (2.2), (2.4), and (2.7) into the second equation of system (1.5) yields

$$q_{\sigma\sigma} = a(\gamma-1)\Big(\theta_\sigma^2 - \frac{aP_0}{R_0}\, \theta_\sigma^{1-\gamma}\, \mathrm{e}^{-q/a}\Big). \tag{2.8}$$

After transformation to the new variables, the third equation in (1.5) becomes

$$\theta_{\sigma\sigma} + \frac{P_0}{R_0}\, \mathrm{e}^{\frac{a+1}{a(\gamma-1)}\, q} \left(\frac{1}{\theta_\sigma^\gamma\, \mathrm{e}^{\frac{a+\gamma}{a(\gamma-1)}\, q}}\right)_\sigma = -\frac{1}{a(\gamma-1)}\, \theta_\sigma q_\sigma. \tag{2.9}$$

After some transformations, Eq. (2.9) reduces to the equation

$$\Big(\theta_\sigma^{\gamma+1} - \frac{\gamma P_0}{R_0}\, \mathrm{e}^{-q/a}\Big)\theta_{\sigma\sigma} - \frac{(a+\gamma)P_0}{a(\gamma-1)R_0}\, \mathrm{e}^{-q/a}\, q_\sigma \theta_\sigma + \frac{\theta_\sigma^{\gamma+2} q_\sigma}{a(\gamma-1)} = 0. \tag{2.10}$$

Thus, the following statement is proved.

Lemma 1. *The factor-system* (1.5) *for solution* (1.4) *reduces to the system of two equations* (2.8) *and* (2.10) *for the two functions θ and q given by formulas* (2.1) *and* (2.3) *and dependent on the variable σ.*

Since Eqs. (2.8) and (2.10) do not explicitly contain the required function θ, their order can be reduced by introducing the new functions $Q = Q(q)$ and $\Theta = \Theta(q)$:

$$q_\sigma = Q(q), \qquad \theta_\sigma = \Theta(q). \tag{2.11}$$

Equations (2.11) lead to the following formulas for the second derivatives:

$$q_{\sigma\sigma} = Q_q q_\sigma = Q Q_q, \qquad \theta_{\sigma\sigma} = \Theta_q q_\sigma = Q\Theta_q. \tag{2.12}$$

Substituting (2.11) and (2.12) into (2.8) and (2.10), we obtain the equations

$$QQ_q = a(\gamma-1)\Big(\Theta^2 - \frac{ac_0^2}{\gamma}\, \Theta^{1-\gamma}\, \mathrm{e}^{-q/a}\Big); \tag{2.13}$$

$$Q\Theta^{1+\gamma}\Theta_q - c_0^2\, \mathrm{e}^{-q/a}\, Q\Theta_q - \frac{(a+\gamma)c_0^2}{\gamma a(\gamma-1)}\, \mathrm{e}^{-q/a}\, Q\Theta + \frac{Q\Theta^{\gamma+2}}{a(\gamma-1)} = 0. \tag{2.14}$$

It should be noted that Eq. (2.14) can be divided by the nonzero multiplier $Q \neq 0$, and, as a result, it becomes an equation for determining the function Θ. Solving this equation, from (2.13) we find the function Q. Thus, the equations of system (1.5) were split and, hence, can be integrated sequentially.

Instead of q, it is convenient to introduce the new independent variable λ:

$$\lambda = \mathrm{e}^{-q/a}. \tag{2.15}$$

Then, $q = -a\ln\lambda$. We note that $\lambda > 0$. The derivatives are recalculated by the formula

$$f_q = -\lambda f_\lambda / a. \tag{2.16}$$

Making the replacement (2.15) in (2.13) and (2.14) and calculating the derivatives by formula (2.16), we obtain the following theorem.

Theorem 1. *The factor-system* (1.5) *of solution* (1.4) *reduces to the system of two first-order equations*

$$\frac{d\Theta}{d\lambda} = \frac{1}{\gamma - 1}\,\frac{\Theta(\Theta^{\gamma+1} - (c_0^2(a+\gamma)/\gamma)\lambda)}{\lambda(\Theta^{\gamma+1} - c_0^2\lambda)}; \tag{2.17}$$

$$(Q^2)_\lambda = \frac{2a^2(\gamma-1)}{\lambda\Theta^{\gamma-1}}\Big(\frac{ac_0^2}{\gamma}\,\lambda - \Theta^{\gamma+1}\Big) \tag{2.18}$$

for the functions $\Theta = \Theta(\lambda)$ *and* $Q = Q(\lambda)$ *given by formulas* (2.11).

3. Transformation and Integration of Eq. (2.17). Let us show that Eq. (2.17) can be integrated. In (2.17), making the replacement of the function by the formula

$$w = \Theta^{\gamma+1}, \tag{3.1}$$

we obtain the equation

$$\frac{dw}{d\lambda} = \frac{\gamma+1}{\gamma-1}\,\frac{w(w - \alpha_0\lambda)}{\lambda(w - c_0^2\lambda)}, \tag{3.2}$$

where

$$\alpha_0 = c_0^2(a+\gamma)/\gamma. \tag{3.3}$$

Equation (3.2) is homogeneous, and the replacement of the required function

$$v = w/\lambda \tag{3.4}$$

reduces it to the equation with split variables.

Substituting (3.4) into (3.2) and performing some transformations, we obtain the equation

$$\lambda\,\frac{dv}{d\lambda} = \frac{2(v^2 + \beta_0 v)}{(\gamma-1)(v - c_0^2)}, \tag{3.5}$$

where the constant $\beta_0 = (c_0^2(\gamma-1) - \alpha_0(\gamma+1))/2$, according to (3.3), has the form

$$\beta_0 = -c_0^2(a + (a+2)\gamma)/(2\gamma). \tag{3.6}$$

The general solution of Eq. (3.5) is represented as the integral

$$\int \frac{(v - c_0^2)\,dv}{v^2 + \beta_0 v} = \frac{2}{\gamma-1}\,\ln\frac{\lambda}{\lambda_0}, \tag{3.7}$$

where λ_0 is the integration constant.

The integral on the left side of formula (3.7) is taken in elementary functions, and the solution of Eq. (3.5) is given by the formula

$$|v + \beta_0|^{1 + c_0^2/\beta_0} v^{-c_0^2/\beta_0} = (\lambda/\lambda_0)^{2/(\gamma-1)}, \tag{3.8}$$

where $\beta_0 \neq 0$. Substituting representation (3.4) into formula (3.8), we obtain the solution in terms of the function w:

$$|w + \lambda\beta_0|^{1 + c_0^2/\beta_0} w^{-c_0^2/\beta_0} = \lambda^{(\gamma+1)/(\gamma-1)}\lambda_0^{-2/(\gamma-1)}. \tag{3.9}$$

We note that, according to (3.6),

$$1 + \frac{c_0^2}{\beta_0} = \frac{a(\gamma+1)}{a(\gamma+1) + 2\gamma}, \qquad \frac{c_0^2}{\beta_0} = -\frac{2\gamma}{a(\gamma+1) + 2\gamma}. \tag{3.10}$$

Substituting expression (3.1) for w and the values of the powers from (3.10) into formula (3.9), we obtain the solution in terms of the function Θ:

$$|\Theta^{\gamma+1} + \lambda\beta_0|^{\frac{a(\gamma+1)}{a(\gamma+1)+2\gamma}} = \lambda_0^{-\frac{2}{\gamma-1}}\lambda^{\frac{\gamma+1}{\gamma-1}}\Theta^{\frac{2\gamma}{a(\gamma+1)+2\gamma}}. \tag{3.11}$$

For $\beta_0 = 0$, which corresponds to the value $a = -2\gamma/(\gamma+1)$, integral (3.7) is simplified and becomes

$$\int \Big(\frac{1}{v} - \frac{c_0^2}{v^2}\Big)\, dv = \frac{2}{\gamma-1} \ln \frac{\lambda}{\lambda_0}.$$

In this case, the solution is given by the formula

$$\ln v + \frac{c_0^2}{v} = \frac{2}{\gamma-1} \ln \frac{\lambda}{\lambda_0}. \tag{3.12}$$

Formula (3.12) can be reduced to the form

$$v\, e^{c_0^2/v} = (\lambda/\lambda_0)^{2/(\gamma-1)}. \tag{3.13}$$

Substitution of the values of v from (3.4) and (3.1) into (3.13) yields

$$\Theta^{\gamma+1} \exp\big(c_0^2 \lambda \Theta^{-(\gamma+1)}\big) = \lambda_0^{-2/(\gamma-1)} \lambda^{(\gamma+1)/(\gamma-1)}. \tag{3.14}$$

Let us formulate the result as the following statement.

Theorem 2. *The general solution of Eq.* (2.17) *is given the formula* (3.11) *for* $\beta_0 \neq 0$ *and by formula* (3.14) *for* $\beta_0 = 0$.

Relations (3.11) and (3.14) are implicit differential equations [11] for the derivatives θ_q (or θ_σ).

4. Sound Characteristics, Integral of Motion, and Vorticity. Let us represent some physical flow parameters corresponding to solution (1.4) in terms of the variables λ and v.

In the description of gas motion, an important role is played by sound characteristics — the surfaces or curves in physical space on which the gas velocity component normal to them is equal to the sound velocity [6, 7]. The sound characteristics specified in the physical plane by the beams $\theta = \theta_0$ (or by half-planes of this form in three-dimensional space) are given by the equation $W^2 = c^2$, which, for solution (1.4), becomes

$$W^2(\theta) = \gamma P(\theta)/R(\theta). \tag{4.1}$$

Substituting representation (2.2) for W and (2.7) for P/R into (4.1), we obtain the equation of the characteristics in the form $\Theta^{\gamma+1}/\lambda = c_0^2$ or [in terms of (3.4)] in the form $v = c_0^2$.

Lemma 2. *The sound characteristics of the equations of gas dynamics for solution* (1.4) *that correspond to the beams* $\theta = \theta_0$ *in the physical plane* (*or to half-planes in the three-dimensional case*) *are specified in the plane* $\mathbb{R}^2(\lambda, v)$ *by the horizontal straight lines*

$$v = c_0^2.$$

As a result, the integral of motion relating the radial coordinate of a gas particle and entropy in solution (1.4) is obtained in terms of the variable λ. The equation of the flow streamlines in the plane $\mathbb{R}^2(y, z)$ has the form

$$\frac{dr}{V} = \frac{r\, d\theta}{W}. \tag{4.2}$$

Substituting the differential $d\sigma = d\theta/W$ from (2.1) into (4.2) and integrating this equation, we obtain

$$\ln \frac{r}{r_0} = \int V(\sigma)\, d\sigma, \tag{4.3}$$

where r_0 is the integration constant that corresponds to the initial position of the gas particle on the streamline. Substituting representation (2.4) for V into (4.3) and transforming to the variable λ [see (2.15)], we obtain the following streamline equation:

$$r = r_0 (\lambda_0/\lambda)^{\gamma-1}. \tag{4.4}$$

The initial position r_0 of the gas particle corresponds to the value λ_0 of the parameter λ characterizing the entropy. Since $\gamma - 1 > 0$, from formula (4.4) it follows that r decreases with increasing λ and increases otherwise.

Let us calculate the vortex (1.7) in terms of the variables v and λ. According to (1.7), we have $\boldsymbol{\omega} = (\omega^1, \omega^2, 0)$ for solution (1.4). In Sec. 2, it was found that, for $W \neq 0$, $U = U_0 = \text{const}$, and, hence, $\omega^2 = r^{-1} U_\theta = 0$. We calculate the first vortex component ω^1 by substituting representations (2.2), (2.4), and (2.11) for the velocity component into (1.7):

$$\omega^1 = \frac{1}{r}(W - V_\theta) = \frac{1}{r}\Big(\theta_\sigma - \frac{q_{\sigma\sigma}}{a(\gamma-1)\theta_\sigma}\Big) = \frac{1}{r\Theta}\Big(\Theta^2 - \frac{(Q^2)_q}{2a(\gamma-1)}\Big). \tag{4.5}$$

The further transformation of (4.5) is performed using Eq. (2.13) and formulas (2.15), (3.1), and (3.4):

$$\omega^1 = \frac{ac_0^2}{\gamma r}\lambda\Theta^{-\gamma} = \frac{ac_0^2}{\gamma r}\lambda^{1/(\gamma+1)}v^{-\gamma/(\gamma+1)}. \tag{4.6}$$

Substituting the value of r from integral (4.4) into (4.6), we finally obtain the following formula for the nonzero vortex component in solution (1.4):

$$\omega^1 = \frac{ac_0^2}{\gamma r_0 \lambda_0^{\gamma-1}}\Big(\frac{\lambda^{\gamma^2}}{v^\gamma}\Big)^{1/(\gamma+1)}. \tag{4.7}$$

Lemma 3. *The nonzero vortex component in solution* (1.4) *in terms of the variables* (λ, v) *is given by formula* (4.7). *The vortex increases in absolute value with increasing* λ *or decreasing* v.

5. Investigation of Singularities of the Solution. The analytical solution (1.4) is given by formulas (3.11) and (3.14). These formulas are bulky and difficult to analyze, and, in addition, they describe the intermediate integrals of solution (1.4) in terms of its first derivatives (2.11).

Important information on solution (1.4) can be obtained by analyzing the singular points and manifolds of Eq. (3.5) [12]. In this case, the solution is obtained as a function of the parameter λ and, hence, as a function of the variable entropy. To determine the dependence of the entropy on the spatial coordinates, it is necessary to solve Eq. (3.14) as a differential equation for the function $\Theta(q)$ given by relation (2.11).

Along with Eq. (3.5), we consider the system of equations corresponding to it:

$$\dot{v} = 2(v^2 + \beta_0 v)/(\gamma - 1), \qquad \dot{\lambda} = \lambda(v - c_0^2), \tag{5.1}$$

where the point denotes derivatives with respect to the parameter.

For any values of β_0, system (5.1) has a singular point O with the coordinates $\lambda_0 = 0$ and $v_0 = 0$. For $\beta_0 > 0$, the point O is the unique singular point; for $\beta_0 < 0$, the singular point P_1 appears at which $\lambda_1 = 0$ and $v_1 = -\beta_0$. We recall that, according to (2.15), $\lambda > 0$ and we consider values $\Theta > 0$; therefore, $v > 0$. Thus, the analysis is performed in the first quarter of the plane $\mathbb{R}^2(\lambda, v)$.

From (3.6), it follows that $\beta_0 > 0$ for $a(\gamma+1) + 2\gamma < 0$. We introduce the characteristic parameter

$$a_* = -2\gamma/(\gamma+1) < 0.$$

Then,

$$\beta_0 > 0 \quad \text{at} \quad a < a_*, \qquad \beta_0 < 0 \quad \text{at} \quad a > a_*. \tag{5.2}$$

Since $\gamma > 1$, the value of a_* does not vanish. From (5.2) it follows that, for negative values of the parameter a, $\beta_0 > 0$, and for $\beta_0 < 0$, the values of a can be both negative [$a \in (a_*, 0)$] and positive ($a > 0$).

From the right side of Eq. (3.5), it follows that as $v \to c_0^2$, the derivative $dv/d\lambda \to \infty$. This implies that the tangent to the integral curves becomes vertical. After Eq. (3.5) is resolved for the derivative $d\lambda/dv$, this derivative will vanish on the straight line $v = c_0^2$. Thus, at the points of the straight line $v = c_0^2$, there may be an ambiguity of the solution in terms of the function $v = v(\lambda)$ or a local extremum of the function $\lambda = \lambda(v)$. The occurrence of these cases depends on the behavior of the integral curves in passing through the straight line $v = c_0^2$.

The integral curves above the straight line $v = c_0^2$ correspond to supersonic gas flow in which the circumferential velocity component exceeds the sound velocity. The integral curves below the straight line $v = c_0^2$ correspond to gas motion with $|W| < c$; however, this motion can be supersonic because its radial velocity component $V \neq 0$.

The Jacobi matrix of the right sides of system (5.1) is equal to

$$J = \begin{bmatrix} 2(2v + \beta_0)/(\gamma - 1) & 0 \\ \lambda & v - c_0^2 \end{bmatrix}. \tag{5.3}$$

Then, the matrix J for $v = 0$ has the eigenvalues

$$k_1 = 2\beta_0/(\gamma - 1), \qquad k_2 = -c_0^2 < 0. \tag{5.4}$$

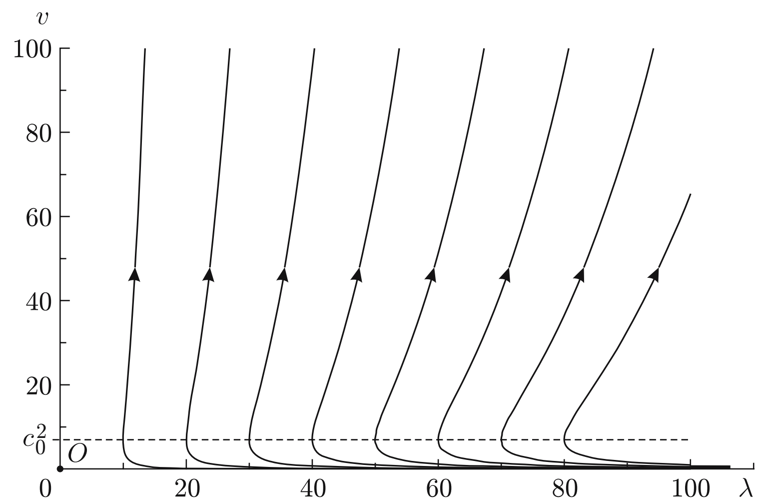

Fig. 1. Integral curves for $\beta_0 > 0$.

At the point P_1, matrix (5.3) has the spectrum ($\beta_0 < 0$)

$$k_1 = -\frac{2\beta_0}{\gamma - 1} > 0, \qquad k_2 = -\beta_0 - c_0^2. \tag{5.5}$$

Thus, three cases are possible.

1. Let $\beta_0 > 0$. Then, $a < a_* < 0$. There is a unique singular point O, at which, according to (5.4), $k_1 > 0$ and $k_2 < 0$. The point O is a saddle, whose separatrices are the coordinate axes; on the λ axis, the gas moves toward the coordinate origin $O(0, 0)$, and on the v axis, in the opposite direction. The phase portrait of the integral curves is presented in Fig. 1. The integral curves intersect the straight line $v = c_0^2$ at a right angle, and two values of v correspond to each value of λ.

According to (2.15), for $a < 0$, the quantities λ and q decrease or increase simultaneously. From (2.3), it follows that, an increase in the entropy S leads to an increase in the quantity q and, hence, in λ in the solutionİn this case, solution (1.4) in the region $v > c_0^2$ can be treated as a supersonic vortex sink. The gas particles start from the characteristic — the straight line $v = c_0^2$ — and move on the integral curves above this straight line. This motion corresponds to an increase in the entropy and [according to (4.4)] to the gas particle motion to the center (coordinate origin). According to Lemma 3, the vorticity of this motion increases.

The integral curves below the straight line $v = c_0^2$ correspond to the gas motion that ceases on the sound characteristic. For this motion, the entropy decreases and the gas particles move away from the coordinate origin, which is a vortex source.

2. Let $\beta_0 < 0$ and $-\beta_0 - c_0^2 < 0$. Then, $a_* < a < 0$. There are two singular points O and P_1. According to (5.4), at the point O, $k_1 < 0$, $k_2 < 0$, i.e., the point O is an attracting node. According to (5.5), at the point P_1 $k_1 > 0$, $k_2 < 0$, i.e., the point P_1 is a saddle. On the separatrix that is the Ov axis, motion starts from the singular point P_1 and the separatrix of the saddle $v = -\beta_0$ enters the singular point P_1. Because, on the segment OP_1 of the Ov axis, $v + \beta_0 < 0$, according to (5.1), it follows that $v < 0$ and motion occurs from the point P_1 to the point O. On the $O\lambda$ axis, motion is also directed toward to the point O. The integral curves for this case are shown in Fig. 2, and the solution is three-valued.

Each value λ correspond to three values of v. Because $a < 0$, as in case 1, the variables λ and S decrease or increase simultaneously. The regions in which the sign of the derivative changes, we denote A, B, and C (see Fig. 2). The solution in the regions B and C coincides with the solution in the case $\beta_0 > 0$. In the region A, the solution is defined for which there are gas expansion (to infinity) and a decrease in the entropy.

3. Let $\beta_0 < 0$ and $-\beta_0 - c_0^2 > 0$. Then, according to (5.3), $a > 0$. From (5.4) and (5.5), it follows that the point O is an attracting node, and, at the point P_1, $k_1 > 0$ and $k_2 > 0$, i.e., it is a repulsing node. The integral curves for this case are shown in Fig. 3.

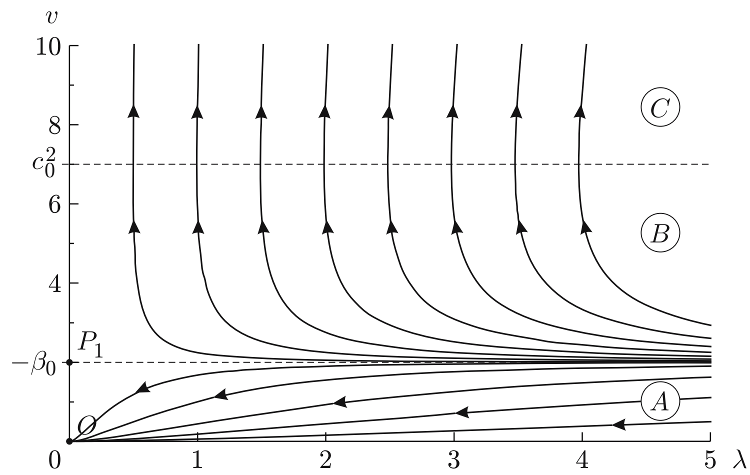

Fig. 2. Integral curves for $-c_0^2 < \beta_0 < 0$.

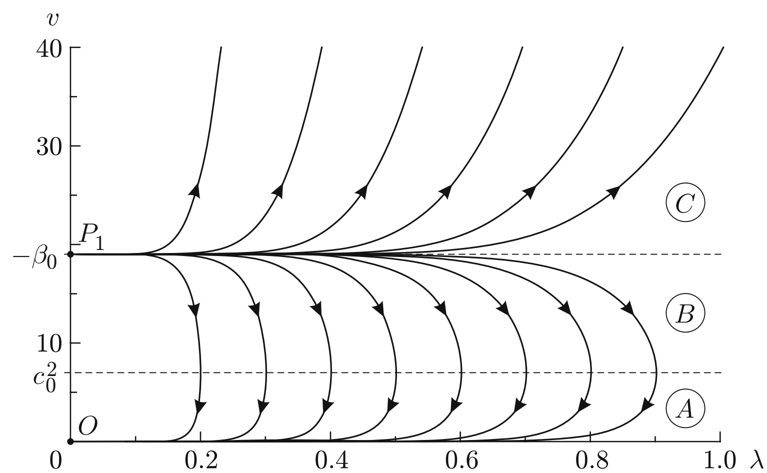

Fig. 3. Integral curves for $\beta_0 < -c_0^2$.

According to formulas (2.3) and (2.15), for $a > 0$, the entropy S increases with decreasing λ. The solution is three-valued, as in case 2.

In the region A, the solution describes the gas flow issuing from the sound characteristic. This solution corresponds to gas expansion to infinity, the radius increases without bound, and the flow entropy also increases. In the region B, the solution describes a supersonic gas sink that ends on the sound characteristic $v = c_0^2$. The solution in the region C corresponds to supersonic gas flow in which the particles start from infinity and gather at the collapse points $\lambda = \pm\infty$ corresponding to the value $r = 0$.

For $\beta_0 = 0$, when $a = a_*$, the system has one degenerate singular point O. At this point, according to (5.4), $k_1 = 0$ and $k_2 < 0$. The description of the behavior of the integral curves in the vicinity of the point O can be obtained by using the methods described in [12] or by using the formula of solution (3.13), which, in this case, has a relatively simple form. In this case, the pattern of the integral curves is similar to that observed in case 1.

Thus, the qualitative analysis of the singularities of Eq. (3.5) provides important information on solution (1.4). The further investigation of this solution should include an analysis of the implicit differential equations (3.11) and (3.14) and investigation of the solution of the system of four equations (2.11), (2.13), and (2.14). This investigation is based on the analysis of the behavior of the solution of Eq. (3.5) performed in the present paper for various values of the parameter a (Fig. 4).

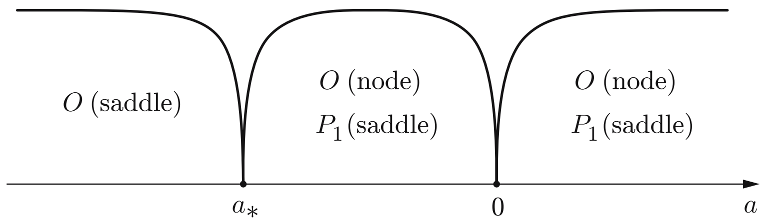

Fig. 4. Types of singular points versus parameter a.

6. Conclusions. Solution (1.4), which is of interest from a point of view of physics, describes not only two-dimensional gas vortices but also spatial twisted gas jets. Indeed, solution (1.4) corresponds to the case $U = U_0 = \text{const}$. For $U_0 \neq 0$, the solution describes three-dimensional motion of a twisted round gas jet at constant velocity U_0 along the Ox axis. The shape of the jet cross section changes: it is enlarged or narrowed according to the description given in Sec. 5. In this case, the sound characteristics $\theta = \theta_0$ are half-planes which start from the Ox axis. Both the regimes of unlimited expansion of the jet to infinity and compression of its collapse to the symmetry axis are possible. For $U_0 = 0$, the gas motion is two-dimensional, and the solution describes two-dimensional gas vortices. For a detailed description and physical treatment of these regimes, it is necessary to analyze the solutions of Eqs. (3.11) or (3.14).

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